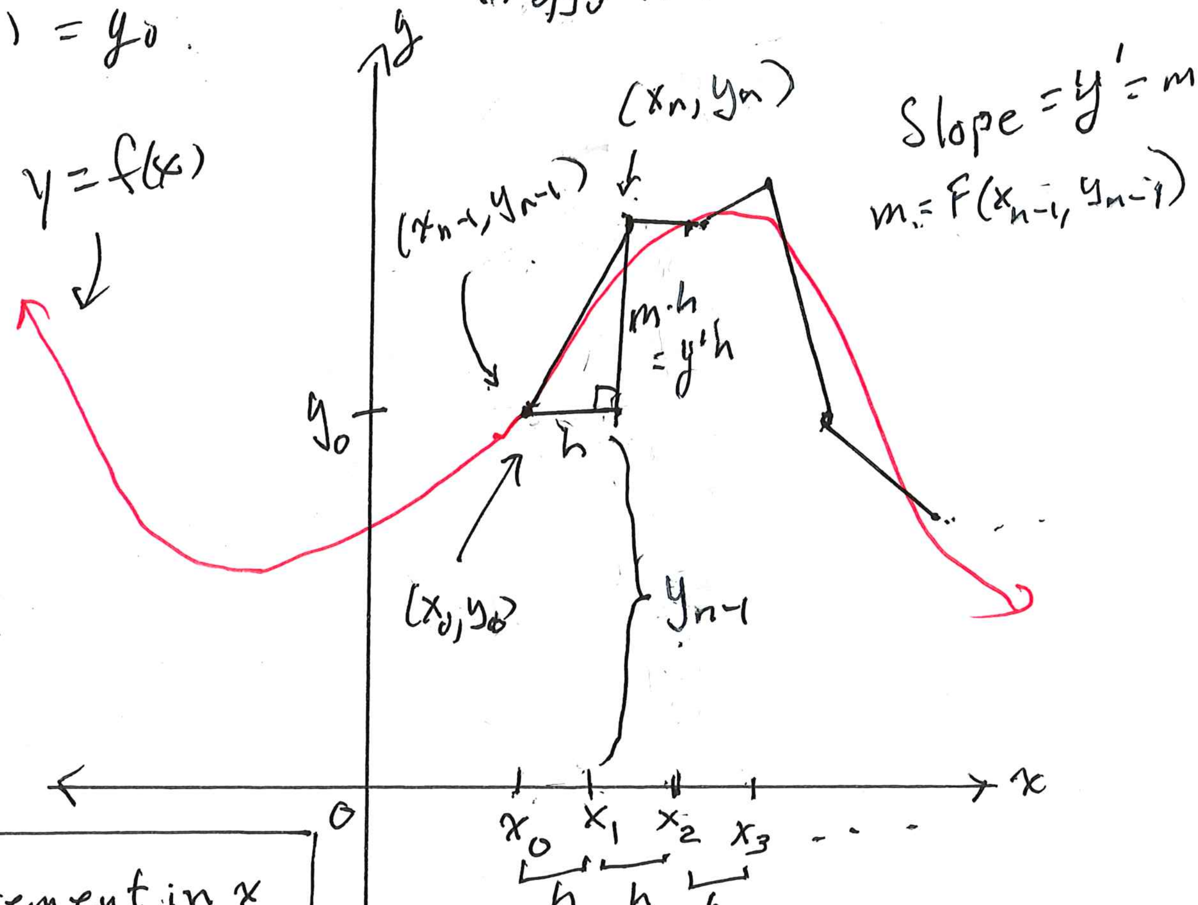
PROJECTED WRITTEN NOTES FROM THE M408D LECTURE ON TUESDAY, MARCH 26, 2024, ON SEC. 9.2 – EULER'S METHOD, and ON SEC. 9.5 – LINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS

CLASS # 19

Euler's Method for Approximating a Solution to AN IVP

IVP: Find $y = f(x) = y(x)$ where $y' = F(x, y)$ and $y(x_0) = y_0$. [(x_0, y_0) is on the solution curve]



h = increment in x

m = slope $= y'$

$y' = F(x, y)$

$x_n = x_{n-1} + h$

$y_n = y_{n-1} + h \cdot \underbrace{F(x_{n-1}, y_{n-1})}_{y'}$

Problem: Use Euler's Method to Approximate a solution $y = f(x)$ for the IVP

$y' = y$ and $y(0) = 1$ using $h = 0.5$

Sol'n: $(x_0, y_0) = (0, 1)$, $y' = y = F(x, y)$

Here: $y' = F(x_{n-1}, y_{n-1}) = y_{n-1}$

$y_n = y_{n-1} + hy'$

n	x_{n-1}	y_{n-1}	$y' = y_{n-1}$	x_n ($x_{n-1}+h$)	$y_n = y_{n-1} + h y_{n-1}$
1	0	1	1	0.5	$1 + (0.5)(1) = 1.5$
2	0.5	1.5	1.5	1.0	$1.5 + (0.5)(1.5) = 2.25$
3	1.0	2.25	2.25	1.5	$2.25 + (0.5)(2.25) = 3.75$
4	⋮	⋮	⋮	⋮	⋮
⋮					

EULER'S METHOD APPLIED TO THE D.E.

$y' = y$, $y(0) = 1$

with Step size $= 0.5$ to Approximate $y(2)$.

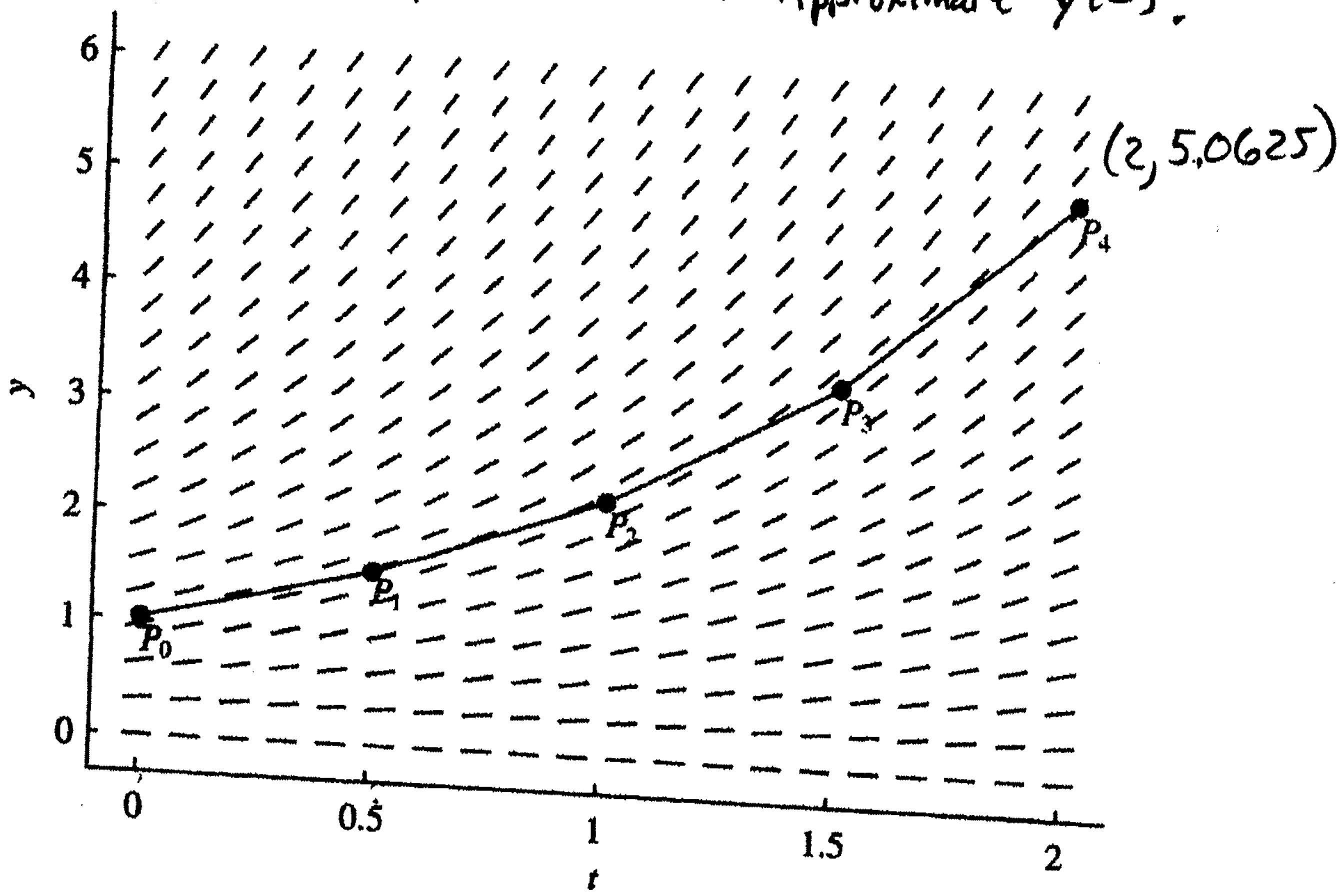


Figure 6 An approximate solution curve of $y' = y$, $y(0) = 1$.

$P_0 = (\ 0\ ,\ 1\ )$

$P_1 = (\ 0.5\ ,\ 1.5\ )$

$P_2 = (\ 1.0\ ,\ 2.25\ )$

$P_3 = (\ 1.5\ ,\ 3.375\ )$

$P_4 = (\ 2.0\ ,\ 5.0625\ )$

The ACTUAL SOLUTION FUNCTION is $y = e^x$

AND $y(2) = e^2 \approx 7.389056$

Actual Solution is $y=e^x$

Using Euler's Method to Approximate y(2)

Differential Equation: y' = y   Initial Values: y(0) = 1

Step Size h = 0.5   y(2) Approx. = 5.0625   Actual y(2) = 7.389056

h	n	x_{n-1}	y_{n-1}	$F(x_{n-1}, y_{n-1})$	x_n	y_n
0.5	1	0	1	1	0.5	1.5
0.5	2	0.5	1.5	1.5	1	2.25
0.5	3	1	2.25	2.25	1.5	3.375
0.5	4	1.5	3.375	3.375	2	5.0625

Using Euler's Method to Approximate y(2)

Differential Equation: y' = y   Initial Values: y(0) = 1

Step Size h = 0.1   y(2) Approx. = 6.7275   Actual y(2) = 7.389056

h	n	x_{n-1}	y_{n-1}	$F(x_{n-1}, y_{n-1})$	x_n	y_n
0.1	1	0	1	1	0.1	1.1
0.1	2	0.1	1.1	1.1	0.2	1.21
0.1	3	0.2	1.21	1.21	0.3	1.331
...	...	...	...	...	...	...
0.1	18	1.7	5.05447	5.05447	1.8	5.559917
0.1	19	1.8	5.559917	5.559917	1.9	6.115909
0.1	20	1.9	6.115909	6.115909	2	6.7275

The actual I.V.P. Solution is $y=e^x$

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Using Euler's Method to Approximate y(2)

Differential Equation: $y' = y$ Initial Values: $y(0) = 1$

Step Size h = 0.01 y(2) Approx. = 7.316018 Actual y(2) = 7.389056

h	n	x_{n-1}	y_{n-1}	$F(x_{n-1}, y_{n-1})$	x_n	y_n
0.01	1	0	1	1	0.01	1.01
0.01	2	0.01	1.01	1.01	0.02	1.0201
0.01	3	0.02	1.0201	1.0201	0.03	1.030301
0.01	4	0.03	1.030301	1.030301	0.04	1.040604
...	...	...	...	...	...	...
0.01	197	1.96	7.030549	7.030549	1.97	7.100855
0.01	198	1.97	7.100855	7.100855	1.98	7.171863
0.01	199	1.98	7.171863	7.171863	1.99	7.243582
0.01	200	1.99	7.243582	7.243582	2	7.316018

Using Euler's Method to Approximate y(2)

Differential Equation: $y' = y$ Initial Values: $y(0) = 1$

Step Size h = 0.001 y(2) Approx. = 7.381676 Actual y(2) = 7.389056

h	n	x_{n-1}	y_{n-1}	$F(x_{n-1}, y_{n-1})$	x_n	y_n
0.001	1	0	1	1	0.001	1.001
0.001	2	0.001	1.001	1.001	0.002	1.002001
0.001	3	0.002	1.002001	1.002001	0.003	1.003003
0.001	4	0.003	1.003003	1.003003	0.004	1.004006
0.001	5	0.004	1.004006	1.004006	0.005	1.00501
0.001	6	0.005	1.00501	1.00501	0.006	1.006015
0.001	7	0.006	1.006015	1.006015	0.007	1.007021
...	...	...	...	...	...	...
0.001	1997	1.996	7.352223	7.352223	1.997	7.359575
0.001	1998	1.997	7.359575	7.359575	1.998	7.366934
0.001	1999	1.998	7.366934	7.366934	1.999	7.374301
0.001	2000	1.999	7.374301	7.374301	2	7.381676

The actual IVP Solution is $y = e^x$.

Solving Some Differential EQUATIONS

A Special Situation

Consider the D.E.

The NICE FORM

$$\ln(x)\cdot\frac{dy}{dx} + \frac{1}{x}y = e^x, \quad x>0.$$

$$\underbrace{[f(x)]\cdot\frac{dy}{dx} + [f'(x)]y}_{} = Q(x)$$

From the product rule of Differentiation

$$[f(x)\cdot y]' = Q(x)$$

$$\int \text{—}\, dx = \int \text{—}\, dx$$

$$\int [f(x)y]'\,dx = f(x)y = \int Q(x)\,dx + C$$

$$\boxed{y = \frac{1}{f(x)}\int Q(x)\,dx + \frac{C}{f(x)}}$$

where C is any real #.

Reconsider $\underbrace{(\ln x)\frac{dy}{dx} + \frac{1}{x}y}_{} = e^x,\ x > 0.$

$$(\ln(x)\cdot y)' = e^x$$

$$\int (\ln(x)\,y)'\,dx = \int e^x\,dx$$

$$\ln(x)\cdot y = e^x + C$$

$$\boxed{y = \frac{e^x}{\ln(x)} + \frac{C}{\ln(x)}}$$

where C is any real #.

Def'n: A LINEAR FIRST-ORDER D.E. is one that can be put in the Standard Form:

$$\frac{dy}{dx} + P(x)\, y = Q(x)$$

Comment: The D.E. from earlier is a Linear 1st-Order D.E.

$$\ln(x) \frac{dy}{dx} + \frac{1}{x} y = e^x$$

$$\frac{dy}{dx} = \left(\frac{1}{\ln x} \cdot \frac{1}{x}\right) y = \frac{e^x}{\ln x}$$

Ex: Not a Linear 1st-Order D.E.:

$$x y' + x^2 y^2 = x^3$$

$$y' + x y^2 = x^2$$

The y^2 factor makes it NOT a Linear 1st-Order D.E.

To Solve a Linear 1^{st} order D.E.,

multiply through by an Integrating FACTOR $I(x)$,

where

$$I(x) = e^{\int P(x)\,dx}$$ ← Set $C = 0$ in the indefinite integral.

$$I'(x) = e^{\int P(x)\,dx} \cdot P(x) = I(x)\,P(x)$$

For $\dfrac{dy}{dx} + P(x)\,y = Q(x)$

The NICE FORM

$$\underbrace{I(x)\frac{dy}{dx} + \overbrace{I(x)\,P(x)}^{I'(x)}y}_{(I(x)\,y)'} = I(x)\,Q(x)$$

$$\int (I(x) \cdot y)'\,dx = I(x)\,y = \int I(x)\,Q(x)\,dx + C$$

$$\boxed{y = \frac{1}{I(x)}\int I(x)\,Q(x)\,dx + \frac{C}{I(x)}}$$

where C is any real #.

It can happen that the last integral has no analytic solution:

Ex: $x^4 y = \int I(x)\, Q(x)\, dx = \int e^{-x^2} dx + C$

$$y = \frac{1}{x^4} \underbrace{\int e^{-x^2} dx} + \frac{C}{x^4}$$ where C is any real #

No Formula for this function

[you may be able to find a PSR for it]

Ex: FIND the General Solution of the DE.

$$\frac{dy}{dx} + \frac{1}{x}y = 2 + \frac{1}{x^2}, \quad x > 0$$

Sol'n: $\frac{dy}{dx} + P(x)y = Q(x), \quad x > 0$

$P(x) = \frac{1}{x}, \quad Q(x) = 2 + \frac{1}{x^2}$

Getting $I(x)$:

$$I(x) = e^{\int P(x)dx} = e^{\int \frac{1}{x}dx} = e^{\ln|x|} \xrightarrow{x>0} = e^{\ln x} = x = I(x)$$

$$x\left(\frac{dy}{dx} + \frac{1}{x}y\right) = x\left(2 + \frac{1}{x^2}\right)$$

$$x\frac{dy}{dx} + 1y = 2x + \frac{1}{x}$$

$$(I(x)y)' = \underbrace{(xy)'}_{} = 2x + \frac{1}{x}$$

$$xy = \int (xy)'dx = \int \left(2x + \frac{1}{x}\right)dx$$

$$xy = x^2 + \ln x + C$$

$$\boxed{y = x + \frac{1}{x}\ln x + \frac{C}{x} \text{ where } C \text{ is any real \#.}}$$

THE "LINEAR FIRST-ORDER DIFFERENTIAL EQUATION INITIAL VALUE PROBLEM PRESENTED IN CLASS"

PROBLEM: SOLVE THE FOLLOWING I.V.P.:

$$\frac{dx}{dt} = 4.5 - \frac{x}{300+2t}, \quad x(0) = 0, \ t \geq 0$$

SOLUTION: CONVERTING this D.E. INTO the STANDARD FORM OF A LINEAR FIRST-ORDER D.E.:

$$\frac{dx}{dt} + \left(\frac{1}{300+2t}\right)x = 4.5, \quad x(0) = 0.$$

$$\frac{dx}{dt} + \left(\frac{1}{300+2t}\right)x = \frac{9}{2}.$$

FORM: $\frac{dx}{dt} + P(t)\cdot x = Q(t)$ is a LINEAR FIRST-ORDER DIFFERENTIAL EQUATION with Integrating FACTOR $I(t) = e^{\int P(t)\,dt}$.

$$\int P(t)\,dt = \int \frac{1}{300+2t}\,dt = \frac{1}{2}\int \frac{1}{u}\,du \quad \left(\begin{matrix} u = 300+2t \\ du = 2\,dt \end{matrix}\right)$$

$$= \frac{1}{2}\ln|u| = \frac{1}{2}\ln(300+2t) = \ln\left((300+2t)^{\frac{1}{2}}\right)$$

(Note: since $t \geq 0$, $|300+2t| = (300+2t)$.)

$$\text{So, } I(t) = e^{\int P(t)\,dt} = e^{\int \frac{1}{300+2t}dt} = e^{\ln\left((300+2t)^{\frac{1}{2}}\right)} = (300+2t)^{\frac{1}{2}} = I(t).$$

To : $\frac{dx}{dt} + \left(\frac{1}{300+2t} \right) x = \frac{9}{2}$, we multiply

to both sides the integrating factor $(300+2t)^{1/2}$,

to get : $(300+2t)^{1/2} \frac{dx}{dt} + (300+2t)^{1/2}(300+2t)^{-1} x = \frac{9}{2}(300+2t)^{1/2}$

$$(300+2t)^{1/2}\frac{dx}{dt} + (300+2t)^{-1/2} x = \frac{9}{2}(300+2t)^{1/2}$$

$$\therefore \quad \left[(300+2t)^{\frac{1}{2}} \cdot x\right]' = \frac{9}{2}(300+2t)^{1/2}$$

$$\therefore (300+2t)^{1/2} \cdot x = \int [(300+2t)^{\frac{1}{2}} \cdot x]' dx = \int \frac{9}{2}(300+2t)^{1/2} dt$$

$$= \left(\frac{9}{2}\right)\left(\frac{1}{2}\right) \int u^{1/2} du \quad \left(\text{where } u = (300+2t) \text{ and } du = 2dt\right)$$

$$= \frac{9}{4} \times \frac{2}{3} u^{3/2} + C$$

$$(300+2t)^{1/2} \cdot x = \frac{3}{2}(300+2t)^{3/2} + C$$

(multiplying both sides by $(300+2t)^{-1/2}$):

$$450 + 3t + C(300+2t)^{-1/2}$$

$$x = \frac{3}{2}(300+2t) + C(300+2t)^{-1/2} = \underline{450 + 3t + C(300+2t)^{-1/2}}$$

when $t=0$, $x=0$ and so, $0 = 450 + 3 \times 0 + C(300)^{-1/2}$

$$0 = 450 + \frac{C}{\sqrt{300}} \Rightarrow \frac{C}{10\sqrt{3}} = -450$$

$$\boxed{C = -4500\sqrt{3}}$$

So, $\boxed{x(t) = 450 + 3t - 4500\sqrt{3}\,(300+2t)^{-1/2}}$